Geometric Characterization and Treatments for Degenerate Cases in Mesh Simplification

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Abstract

In the edge contraction algorithm for surface simplification, two major steps are (1) ordering of the edges to be collapsed, and (2) positioning of the new vertex to replace the collapsed edge. The accuracy of the simplified result greatly depends on which procedures are taken in the above steps. During the simplification process, degenerate cases can occur in which the optimization problem for the new vertex position produces multiple solutions. In the previous algorithms, such redundancy was utilized to achieve other desirable effects (e.g, increasing the regularness of the triangles). In this paper we give a complete characterization of the geometric situation for the degenerate cases, and show that performing edge contraction procedure in those cases can produce unexpected damage to the original shape. To cope with the problem, we propose a new procedure, called edge elimination, *which eliminates the degenerate edge and all the adjacent ones, and then retriangulates the hole in a way the geometrical shape remains unchanged. The new algorithm proceeds like the edge contraction algorithm. Whenever it has to process a degenerate edge, however, the edge elimination procedure is performed instead of edge contraction.*

Keywords. edge elimination, degenerate cases, edge contraction, mesh simplification, optimization.

1 Introduction

In real-time applications models with millions of polygons are burdensome even though contemporary graphical devices are high-performance. Therefore simplification of surfaces has been the subject of a great deal of research. Simplification algorithms can be divided into three main categories according to the strategies employed [5]; vertex clustering [12, 13], vertex decimation [1, 3, 15, 16], and edge contraction [2, 5, 6, 8, 9, 11, 14].

Edge contraction plays an important role in surface simplification. Each algorithm in this category replaces an edge in order of importance with a vertex, removing two triangles shared the edge, effecting the reduction of three edges and one vertex (see Figure 1 (a)). In this process it is not necessary to retriangulate the surface. However, after removing edge a new vertex has to be introduced and placed so as not to degrade the quality of the shape, which can be measured by distance between the old and the new surfaces or by volume changes. One is to minimize distances from a set of planes containing the nearby triangle to an object vertex [5, 14]. The other is to control the local volume of the model to preserve the volume of the original model [6]. In many widely-used algorithms for surface simplification, it is required to solve linear systems derived from the minimization of the cost functions.

Solving the system of linear equations $Ax = b$ is requested from almost every science and engineering field. Mesh simplification has no exception. To simplify a given mesh it is required to build a cost function which is used in ordering edges and in locating vertices newly introduced after contracting edges. In general the cost function is converted into a linear system. For instance, when volume optimization procedure is employed the matrix A is of the form $\sum N_j^T N_j$ where the summation is taken over all neighboring edges and N_j is a normal vector to triangular surfaces (see (2)). The vector b is the sum of all normal vectors N_j to the triangles neighboring target edge each of which is weighted by the volume of the triangles neighboring target edge. The linear system is in trouble when $\det A$ is zero. This paper gives a theoretical characterization of the degenerate cases in distance or volume optimization [5, 11], i.e, the cases when $\det A = 0$ and hence the solution space forms a line or a plane. At a first glance, it does not seem to be a problem: one may pick a point on the line or plane and utilize the redundancy to achieve other useful goals. In fact, this approach was taken in the previous edge contraction algorithms.

In this paper, we show that if the determinant is zero the geometry around the troublesome edge has to be a certain shape. A further reasoning on the shape surprisingly reveals that the edge contraction procedure can make a damage to the original shape, no matter which point is taken from the solution space. For the degenerate cases, we propose a new procedure called *edge elimination* that utilizes the geometrical information to avoid unnecessary shape changes. Here we concentrate on volume optimization. Recently, Kim *et al.* [10] showed that volume optimization is distance optimization weighted by the area of triangles adjacent to the contracted edge. Thus edge elimination is also applicable to distance optimization.

The subsequent sections are organized as follows: Section 2 gives a brief review on the previous work. Section 3 provides a complete geometric characterization of the degenerate cases. In Section 4 we develop the edge elimination procedure and give a new ordering algorithm that reflects the changes due to the edge elimination. Finally, Section 5 concludes the paper.

2 Related Work

The algorithm based on iterative edge contractions is getting attention these days due to its local control and simplicity. The algorithm is naturally formulated into an optimization problem. Hoppe *et al.* [9] minimized a global energy function which is the sum of distance, representation, and spring energy terms. The distance energy term measures the total squared distance of the points from the mesh,the representation energy term penalizes the number of vertices, and the spring energy term is introduced to regularize the optimization problem. In [8] Hoppe extended the above work to include the scalar and discontinuity energy terms. He stored a sequence of simplification operations to construct a continuous-resolution representation. Ronfard and Rossignac [14] minimized the maximum of distances between a vertex of the simplified model and the planes containing the triangles adjacent to the collapsed edge. Instead of the maximum distance, Garland and Heckbert [5] used the sum of squared distances. Their algorithm maintains surface error approximations using quadric matrices and is able to join unconnected regions of models. A 4×4 matrix Q is associated with each vertex v. The error at the vertex is defined to be vQv^T, and when a pair is contracted, their matrices are added together to form the matrix for the resulting vertex. Cohen *et al.* [2] presented an algorithm which computes a piecewise linear mapping between the original and simplified models. They chose a vertex in two dimensions and completed the edge contraction by minimizing the error in the remaining third direction. Guéziec [6] proposed the new vertex should be placed so that the volume of the original model is preserved. He used the error radii at each vertex to bound the simplification error. Later Lindstrom and Turk [11] extended Guéziec's work: they added boundary optimization to deal with the boundaries and triangle shape optimization to avoid singularity by placing the new vertex nearby the center of surrounding triangles.

There are also other simplification algorithms. Schroeder *et al.* [15] proposed the decimation method that removes vertices iteratively and retriangulates the surrounding vertices. Hamann [7] made use of the local curvature information together with equi-angularity, so that triangles in a nearly planar surface and slivers are removed first. Rossignac and Borrel [13] proposed the clustering algorithm in which the vertices close to each other are merged into one vertex. The method is fast but the quality of simplification is relatively low.

3 Geometric Characterization of Degenerate Cases

Let us first define a few notations. For an edge e, let v_1^e and v_2^e be the two vertices. For a vertex v, let $I(v)$ represent the set of indices of the triangles adjacent to v, and $I(e) = I(v_1^e) \cup I(v_2^e)$. Let v_{j1}, v_{j2}, and v_{j3} be the vertices of a triangle T_j. In the following, all the vectors are row vectors.

Lindstrom and Turk [11] proposed an edge contraction algorithm (see Figure 1 (a)) that tries to minimize the change in volume. Let $\bar{v}$ be the new vertex that replaces the collapsed edge and $V_j(\bar{v}) = V(\bar{v}, v_{j1}, v_{j2}, v_{j3})$ be the volume of the tetrahedron consisting of the vertex $\bar{v}$ and a trian-

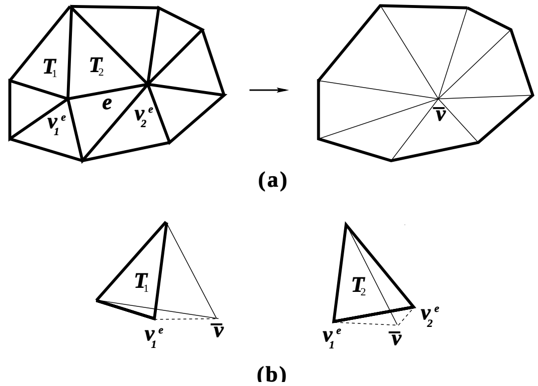

Figure 1. Edge contraction. (a) Edge e is contracted into a vertex $\bar{v}$. (b) The two tetrahedra formed by (T_1,$\bar{v}$) and (T_2,$\bar{v}$) represent the volume change at T_1 and T_2, respectively, due to the procedure.

gle T_j as shown in Figure 1 (b). To consider the variance of the volume differences during the contraction process each V_j is squared to obtain [11]

$$V_e(\bar{v}) = \bar{v} \sum_{j\in I(e)} (N_j^T N_j)\bar{v}^T - 2\sum_{j\in I(e)} \det(v_{j1}, v_{j2}, v_{j3})\, N_j \cdot \bar{v} + \sum_{j\in I(e)} \det(v_{j1}, v_{j2}, v_{j3})^2 \tag{1}$$

where $N_j = (v_{j1} - v_{j2}) \times (v_{j1} - v_{j3})$. In volume optimization a new vertex is sought to minimize $V_e(\bar{v})$. If each factor in (1) is divided by $\|N_j\|^2$, (1) becomes the objective function of distance optimization [5]. (Refer to [10] for details.) If we set

$$A = \sum_{j\in I(e)} N_j^T N_j \tag{2}$$

then the problem is reduced to finding the solution of the system

$$A\bar{v} = \sum_{j\in I(e)} \det(v_{j1}, v_{j2}, v_{j3}) N_j. \tag{3}$$

Let us consider the case when the determinant of A vanishes or equivalently the nullity of A is strictly positive. Then the solution set of the system (3) has to contain a line and so we may write the minimizer of (1) as $v(t) = tu + w$ for $t \in \mathbb{R}$ for some vectors u and w. Then (1) can be rewritten as

$$F(t) = \sum_{j\in I(e)} \|N_j\|^2 \left\{n_j \cdot (tu + w) + c_j\right\}^2$$

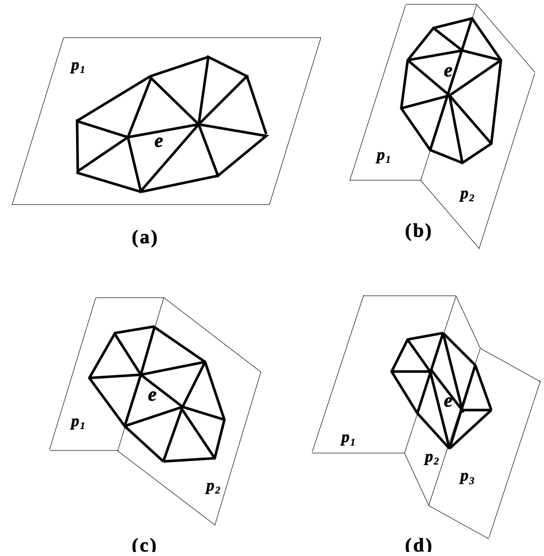

Figure 2. (a) $\cup_{j\in I(e)} T_j$ forms a single plane p_1. (b), (c) $\cup_{j\in I(e)} T_j$ forms two planes p_1 and p_2. (d) $\cup_{j\in I(e)} T_j$ forms three planes.

where n_j is an outward unit normal vector of T_j and c_j is the signed distance between the origin and the plane containing the triangle T_j. Note that the sign of c_j is positive when n_j faces the origin. Since $F'(t) = 0$, we have

$$\sum_{j\in I(e)} \|N_j\|^2 (n_j \cdot u)^2 = 0$$

and hence $n_j \cdot u = 0$ for all $j \in I(e)$. This means that the solution line $v(t)$ is parallel to T_j for all $j \in I(e)$. Let E be the set of edges in $\cup_{i\in I(e)} T_j - \partial(\cup_{j\in I(e)} T_j)$, where ∂ denotes the boundary. We note that at most four edges in $E - \{e\}$ can be parallel to the line $v(t)$ since every edge in $E - \{e\}$ shares a vertex with the edge e. In the following proposition, we show that if the determinant of A is zero, there is an interesting regularity among the triangles in $\cup_{j\in I(e)} T_j$.

Proposition 1. *If* $\det A = 0$, *then the triangles in* $\cup_{j\in I(e)} T_j$ *form one, two, or at most three planes.*

Proof. It is sufficient to consider the following four cases: (a) no edge or one edge, (b) two edges, (c) three edges, (d) four edges in E are parallel to $v(t)$.

Case (a): If only one edge $\tilde{e} \in E$ is parallel to $v(t)$, then for any other edge $e' \neq \tilde{e}$ in E, two triangles sharing e' should be on the same plane since they are both parallel to $v(t)$. In this way we can show that all the edges in E are contained in one plane (see Figure 2 (a)). If no edge is parallel to $v(t)$, by a similar argument, $\cup_{j\in I(e)} T_j$ forms a single plane.

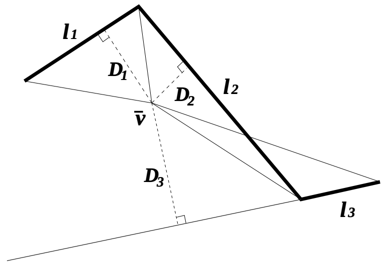

Figure 3. The minimization problem for the case of Figure 2 (d) can be converted into an equivalent two-dimensional problem.

Case (b): When the edges e_1 and e_2 in E that are parallel to $v(t)$ are linked, and both of them intersect $\partial(\cup_{j\in I(e)} T_j)$, then $\cup_{j\in I(e)} T_j$ constitutes one plane or two planes as shown in Figure 2 (c). In the other cases of Case (b), $\cup_{j\in I(e)} T_j$ forms a single plane.

Case (c): When the three edges that are parallel to $v(t)$ form a single line, $\cup_{j\in I(e)} T_j$ forms one or two planes (Figure 2 (b)). If the three edges do not form a single line, then the case is similar to Case (b).

Case (d): In this case, $\cup_{j\in I(e)} T_j$ constitutes one, two, or three planes as in Figure 2 (d). □

Now let us see where the position of the new vertex $\bar{v}$ should be for the above cases. When $\cup_{j\in I(e)} T_j$ constitutes a single plane p_1 (Figure 2 (a)), the solution is the plane p_1 itself and $\bar{v}$ can be placed anywhere on the plane p_1 but inside $\partial(\cup_{j\in I(e)} T_j)$ without modifying the geometrical shape. If $\cup_{j\in I(e)} T_j$ constitutes two planes p_1 and p_2 (Figure 2 (b) or (c)), then the solution is $p_1 \cap p_2$ and $\bar{v}$ can be placed anywhere on the segment $p_1 \cap p_2 \cap (\cup_{j\in I(e)} T_j)$. The case when $\cup_{j\in I(e)} T_j$ constitutes three planes p_1, p_2, and p_3 (Figure 2 (d)), however, needs a special attention. Since p_i's are parallel to $v(t)$, we can reduce the problem to two dimensions. Let $J_i = \{k \in I(e) : T_k \in p_i\}$ and $l_i = \sum_{k\in J_i} \text{area}\,(T_k)$. Then the objective function (1) of the edge contraction procedure becomes $\sum_{i=1}^{3} l_i^2 D_i^2$ (see Figure 3), where l_i and l_{i+1} make the same angle as p_i and p_{i+1} ($i = 1, 2$), and D_i denotes the distance between $\bar{v}$ and a line containing l_i. The above means that the problem is equivalent to minimizing the sum of squared-areas of the three triangles formed by $(l_i, \bar{v})$ ($i = 1, 2, 3$). But a further observation on Figure 3 tells us that, no matter where $\bar{v}$ is placed, distortion in the simplified model is inevitable. Therefore we conclude that edge contraction is not the right procedure to be taken when $\cup_{j\in I(e)} T_j$ constitutes three planes. Instead we propose that the *edge elimination* procedure should be performed for such special cases, which perfectly preserves the original shape by utilizing the local geometry information.

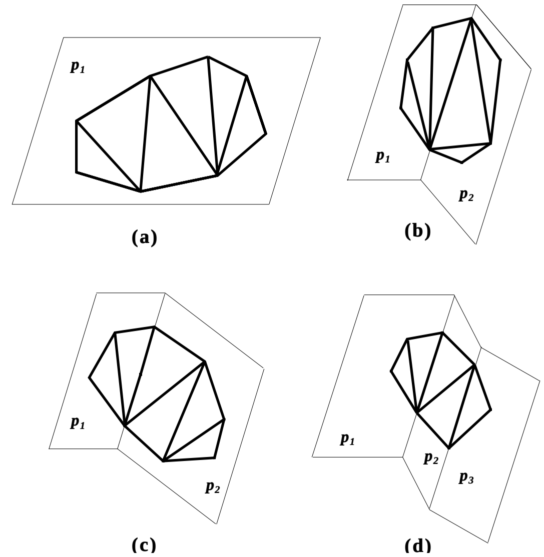

Figure 4. Edge elimination: Each figure shows the result of edge elimination applied to the corresponding cases in Figure 2. Note that the original geometrical shape is always preserved after the procedure.

Details of the new procedure is explained in the next section.

4 Edge Elimination

In the previous section we have investigated the cases when the solution of the minimization problem forms a line or a plane. In the cases $\cup_{j\in I(e)} T_j$ constitutes one or two planes, $\Omega = \{v_1^e, v_2^e, (v_1^e + v_2^e)/2\}$ always contains a reasonable solution and we can choose any one among them without loss of any big significance. Actually it is the procedure which has been widely used. However in the case when $\cup_{j\in I(e)} T_j$ constitutes three planes p_1, p_2 and p_3, we have shown that Ω does not contain a solution. In [11], when volume optimization under the constraint of volume preservation does not yield a single solution, triangle shape optimization is done in order to place the new vertex nearby the center of surrounding triangles.

Due to the investigation in the previous section, we now know the local geometry explicitly, and that simply following the conventional edge contraction procedure for the edge e can damage the original shape. Let us take advantage of this information in devising the new procedure for the degenerate cases.

In edge elimination, we remove all the edges in E first such that all interior vertices in $\cup_{j\in I(e)} T_j$ are discarded. The resulting hole is retriangulated so that the original shape remains unchanged. Figure 4 shows the examples. Triangulation is not the subject we are going to discuss further in the paper. Interested readers may refer to [4].

In addition to the accurate shape preservation, the procedure has another advantage in the reduction of vertices and edges. Suppose that the hole $\partial(\cup_{j\in I(e)} T_j)$ is an n-gon on a closed manifold surface. Then the original model has $n+2$ triangles, $2n+3$ edges and $n+2$ vertices. After an edge contraction operation, it is reduced to n triangles, $2n$ edges, and $n+1$ vertices. Therefore the operation achieves reduction of two triangles, three edges, and one vertex. On the other hand, an edge elimination operation achieves reduction of *four* triangles, *six* edges, *two* vertices, and hence reduces two more triangles, three more edges, and one more vertex than those of the edge contraction. The comparison of the two methods is summarized in Table 1.

When $\cup_{j\in I(e)} T_j$ constitutes one or two planes, the edge contraction and elimination do not make any difference in the resulting geometrical shape. But the edge elimination out-performs the edge contraction in the amount of edge and vertex reduction. Therefore we suggest the edge elimination be used whenever the determinant is zero.

In simplification algorithms based on edge contraction, the result is greatly affected by the order in which the edges are collapsed. Note that our algorithm is basically an edge contraction algorithm, except that the edge elimination procedure is done when A is singular.

Conventional edge contraction algorithm finds a point for each edge that minimizes a cost function [5, 2, 11]. Then the edges are sorted according to the associated minimum values. Thus the edge with the smallest minimum value is contracted first. The cost function basically measures the error between the original model and the simplified model during the contraction.

Let us consider the case when the determinant of the matrix A is zero and $\cup_{j\in I(e)} T_j$ forms three planes. In such a case, edge elimination produces no shape change. Therefore the value of the cost function should be zero. But the value evaluated by the conventional edge contraction algorithm will not be zero according to the discussion given in Section 3 (Figure 3). That means the old cost function has to be modified. As for the cases when A is singular and $\cup_{j\in I(e)} T_j$ constitutes one or two planes, the value of the cost function is zero whether edge contraction or edge elimination is used. Let us call the edges for which $\det A$ is zero the *degenerate edges*.

Based on the above discussion, the new ordering scheme we propose here is that the degenerate edges should be processed before the other edges since it does not induce any shape change. Even though we can randomly remove the degenerate edges, we can order them for the regularity of the mesh. For example, to preserve the regularity of the triangle size it is possible to order the lengths of the degenerate edges.

Furthermore, one advantage of edge contraction algorithm is that it is possible to incrementally compute function values after edge contraction only in the vicinity of the contracted edge. The proposed edge elimination scheme also inherits this advantage.

Table 1. Reduction of triangles, edges, and vertices in edge contraction and edge elimination.

	contraction	elimination
# triangles	2	4
# edges	3	6
# vertices	1	2

5 Conclusion

In this paper we investigated the degenerate cases that can arise during the process of solving an optimization problem in edge contraction, especially for the distance and volume optimization [5, 11]. In those cases the solution space forms a line or a plane. In the conventional edge contraction algorithms such redundancy is exploited to achieve triangle shape optimization or other useful goals.

We gave a complete characterization of the geometrical situation for the triangles adjoining the troublesome edge e in Section 3. We proved that they can form three planes at most. Due to this fact, it can be shown that the conventional edge contraction can produce unnecessary damages to the original model (see Figure 3). In Section 4 we presented a new procedure for the above degenerate cases, the *edge elimination* that perfectly preserves the original shape. The procedure has another benefit: it reduces more edges and vertices than the edge contraction operation. Also, we proposed a new ordering scheme in which all the degenerate edges are processed before the other edges. Volume optimization is similar to distance optimization, so it is possible to develop edge elimination for distance optimization.

Our work is rather theoretical. It deals with the case when the determinant of A in (2) is zero. It does not deal with, however, the cases when the determinant is near zero. It is another important step that needs to be studied to apply the edge elimination procedure in real problems. We need to find out how well the edge elimination process will preserve the local geometry compared to the edge contraction or other algorithms when the determinant is near zero.

We are currently investigating the "near-zero" cases and implementing mesh simplification based on the new observation.

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